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Long-Range Order in Nonequilibrium Systems of Interacting Brownian Linear Oscillators

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Long-range order (lro) is established with the help of a generalized Peierls argument for non-equilibrium lattice systems of one-dimensional (linear) interacting oscillators whose equation of motion (for a finite number of them) is the Smolouchowski equation for the density of a probability distribution. Interaction is mediated through the pair nearest-neighbor quadratic translation invariant potential. The initial density is Gibbsian with a potential energy satisfying the Ruelle superstability and regularity conditions.

KEY WORDS: Non-equilibrium long-range order; contour and superstability bounds.

1. INTRODUCTION

Dynamics in the system of finite number of Brownian linear oscillators, whose one-dimensional coordinates q_x are indexed by a site x of the d-dimensional lattice $\mathbb{Z}^d$, interacting via a pair potential $(q_x - q_y)^2$, is governed by the Smoluchowski equation for the density $\rho^0(q_\Lambda; t)$ of a probability distribution, $q_\Lambda = (q_x, x \in \Lambda) \in \mathbb{R}^{|\Lambda|}$, Λ is a hypercube in $\mathbb{Z}^d$ ($|\Lambda|$ is the cardinality of Λ).

$$\frac{\partial}{\partial t} \rho^0(q_\Lambda; t) = \sum_{x \in \Lambda} \partial_x \{\partial_x + \beta\, \partial_x U_g(q_\Lambda)\}\, \rho^0(q_\Lambda; t), \tag{1.1}$$

$$U_g(q_\Lambda) = \sum_{x \in \Lambda} u_g^0(q_x) + \sum_{\langle x, y\rangle \in \Lambda} (q_x - q_y)^2, \tag{1.2}$$

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where β is the inverse temperature, $\partial_x = \frac{\partial}{\partial q_x}$,

$$u_g^0(q) = \eta g^{-n} q^{2n} - q^2, \qquad n \in \mathbb{Z}^+, \quad n > 1, \quad \eta > 0,$$

$\langle x, y \rangle$ means that x, y are (n-n) nearest neighbors and g is the parameter that determines the depth of two symmetric wells of the potential u_g^0. This parameter can be interpreted as the strength of the n-n interaction: rescaling the oscillator variables $q_x \to \sqrt{g}\, q_x$ and the time $t \to g^{-1} t$ we'll derive the Smoluchowski equation with the rescaled potential energy $U(q_\Lambda) = U_g(\sqrt{g}\, q_\Lambda)$ considered in refs. 1 and 2. After the rescaling the time $t \to \beta t$ we'll obtain, also, the Smoluchowski equation with the diffusion coefficient β^{-1}, i.e., with the coefficient before the first term in the bracket in (1.1), and the second term in the bracket without the coefficient β.

We deal with the free boundary condition. The choice of other boundary conditions may be an obstruction for proving an existence of the lro (see remarks in ref. 2).

A simple check shows that the Gibbs distribution $\exp\{-\beta U_g(q_\Lambda)\}$ is the stationary state for (1.1) and that the law of conservation of probability holds for it. It is expected that stationary states for the considered non-equilibrium systems are Gibbsian.

(1.1) is the forward Kolmogorov equation for the stochastic equations

$$\dot{q}_x(t) = -\beta\, \partial_x U_g(q_\Lambda(t)) + \dot{w}_x(t), \qquad x \in \Lambda$$

where $\dot{w}_x(t)$ are independent processes of white noise.

Solutions of the infinite system were proven to exist in ref. 3. A convergent (high temperature) cluster expansion for the associated measures corresponding to Gibbsian initial measures is proposed in ref. 4. In ref. 5 the nonequilibrium systems are treated as Gibbs path systems (see also ref. 6) and an absence of phase transitions is proven if in the initial Gibbs state they don't occur.

Let's consider the nonequilibrium correlation functions assuming that initial correlation functions are Gibbsian and generated by the potential energy U^1

$$\rho^\Lambda(q_X; t) = Z_\Lambda^{-1} \int \rho^0(q_\Lambda; t)\, dq_{\Lambda \setminus X}, \qquad Z_\Lambda = \int \rho^0(q_\Lambda; t)\, dq_\Lambda,$$

where the integrations are performed over $\mathbb{R}^{|\Lambda \setminus X|}$ and $\mathbb{R}^{|\Lambda|}$, respectively,

$$U^1(q_\Lambda) = \kappa U_g(q_\Lambda) + U^0(q_\Lambda), \qquad U^0 \geqslant 0, \quad \kappa > \tfrac{1}{2} \tag{1.3}$$

and U^0 is a translation invariant function which depends on squares of differences of variables.

After the substitution

$$\rho^0(q_\Lambda; t) = e^{-\frac{\beta}{2} U_g(q_\Lambda)} \psi(q_\Lambda, t)$$

the following heat equation for ψ is obtained

$$\frac{\partial}{\partial t} \psi(q_\Lambda; t) = \sum_{x \in \Lambda} \partial_x^2 \psi(q_\Lambda; t) + \beta V(q_\Lambda)\, \psi(q_\Lambda; t), \tag{1.4}$$

$$V(q_\Lambda) = \frac{1}{2} \sum_{x \in \Lambda} \left[-\partial_x^2 U_g(q_\Lambda) + \frac{\beta}{2} (\partial_x U_g(q_\Lambda))^2 \right].$$

The solution of an initial value problem for (1.4) is given by the FK formula.[7] As a result

$$\rho^\Lambda(q_X; t) = \int \rho^\Lambda(\omega_X)\, P_{q_X}(dw_X), \qquad \rho^\Lambda(\omega_X) = Z_\Lambda^{-1} \int e^{-\beta U(\omega_\Lambda)} P_0(d\omega_{\Lambda \setminus X}), \tag{1.5}$$

where $\omega = (q, w) \in \mathbb{R} \times \Omega = \Omega_*$, Ω is the probability space of one-dimensional Wiener paths, starting from the origin, $w \in \Omega$, $P_q(dw)$ is the Wiener measure, $P_0(d\omega) = dq\, P_q(dw)$,

$$Z_\Lambda = \int e^{-\beta U(\omega_\Lambda)} P_0(d\omega_\Lambda), \qquad P_0(d\omega_X) = \prod_{x \in X} P_0(d\omega_x).$$

$$U(\omega_\Lambda) = \tfrac{1}{2} U_g(q_\Lambda) - \tfrac{1}{2} U_g(w_\Lambda(t)) + U^1(w_\Lambda(t)) + \int_0^t V(w_\Lambda(\tau))\, d\tau. \tag{1.6}$$

So, we have to deal with the Gibbs path system characterized by the potential energy $U(\omega_X)$

$$U(\omega_\Lambda) = \sum_{x \in \Lambda} u(\omega_x) + \sum_{\langle x, y \rangle \in \Lambda} \varphi(\omega_x, \omega_y) + U'(\omega_\Lambda), \tag{1.7}$$

where $\langle ., . \rangle$ means nearest neighbors,

$$U'(\omega_\Lambda) = U^0(w_\Lambda(t)) + \beta \int_0^t V'(w_\Lambda(\tau))\, d\tau,$$

$$V'(q_\Lambda) = 4 \sum_{X_3 \in \Lambda,\, |x_1 - x_j| = 1} (q_{x_1} - q_{x_2})(q_{x_1} - q_{x_3}), \qquad X_3 = (x_1, x_2, x_3),$$

$$u(\omega) = \frac{1}{2} u_g^0(q) + \left(\kappa - \frac{1}{2} \right) u_g^0(w(t)) + \frac{1}{2} \int_0^t \left[-\partial^2 u_g^0(w(\tau)) + \frac{\beta}{2} (\partial u_g^0(w(\tau))^2 \right] d\tau,$$

$$\varphi(\omega, \omega') = \left[\frac{1}{2}(q-q')^2 + \left\{\kappa - \frac{1}{2}\right\}(w(t)-w'(t))^2\right] + \beta \int_0^t \phi(w(\tau), w'(\tau))\, d\tau,$$

$$\phi(q_x, q_y) = (\partial u_g^0(q_x) - \partial u_g^0(q_y))(q_x - q_y).$$

Here we took into account that every internal (boundary) lattice site has $2d(2d-1)$ nearest neighbors ($\partial\Lambda$ is the boundary of Λ) and the following relations

$$\partial_x U_g(q_\Lambda) = \partial_x u_g^0(q_x) + 2 \sum_{y \in \Lambda,\, |x-y|=1} (q_x - q_y)$$

$$\sum_{x \in \Lambda} \partial_x^2 U_g(q_\Lambda) = \sum_{x \in \Lambda} \partial_x^2 u_g^0(q_x) + 4d\,|\Lambda| - 2\,|\partial\Lambda|.$$

$$\sum_{x \in \Lambda} (\partial_x U_g(q_\Lambda))^2 = \sum_{x \in \Lambda} (\partial_x u_g^0(q_x))^2 + \sum_{\langle x, y\rangle \in \Lambda} \phi(q_x, q_y) + V'(q_\Lambda),$$

Prime will not denote differentiation in what follows. The condition $\kappa > \frac{1}{2}$ guarantees that that U satisfies the Ruelle superstability bound.

The interaction part of the potential energy is the sum of the translation invariant positive (ferromagnetic) term U', the ferromagnetic term, generated by the pair positive potential $\varphi^+ = \varphi - \varphi^-$ and the non-ferromagnetic term expressed through the pair non-positive potential φ^-

$$\varphi^-(\omega, \omega') = \beta \int_0^t \phi(w(\tau), w'(\tau))\, d\tau,$$

The potential energy U has ferromagnetic ground states if $\kappa \geqslant \frac{1}{2}$, but we can not prove that nonequilibrium "spin" averages satisfy the first Griffiths inequality. This inequality holds in some nonequilibrium spin systems (lattice systems of interacting markovian processes).[8]

Our goal is to prove an existence of the ferromagnetic lro (long-range order) in our systems for sufficiently large g.

The proof follows all the steps of our previous papers[1, 2] devoted to an existence of the ferromagnetic lro in classical and quantum Gibbs oscillator systems in which interaction potential energies are translation invariant. It is possible since reduced density matrices of the quantum systems are expressed in terms of correlation functions of Gibbs path systems with closed paths (see (3.1) and (3.2) in ref. 2).

Our method in refs. 1 and 2 is a general version of the Peierls argument based on the Ruelle superstability bound for the correlation functions. This method is developed here by an application of a new superstability bounds

(1.12), (2.4) for correlation functions in which the ferromagnetic part of the path "potential energy" plays a significant role.

By $\langle\cdot\rangle_\Lambda$, $\langle\cdot\rangle$ we'll denote the average associated to the correlation functions ρ^Λ and their thermodynamic limit ρ, respectively

$$\langle F_X\rangle_\Lambda = \int F_X(q_X)\,\rho^\Lambda(q_X;t)\,dq_X, \qquad \langle F_X\rangle = \int F_X(q_X)\,\rho(q_X;t)\,dq_X.$$

The generalized Peierls argument is formulated in the following lemma (see ref. 9).

Lemma 1.1. Let $\chi^+(\chi^-)$ be the characteristic functions of positive half-line (negative half-line), $\chi_x^+(q_\Lambda) = \chi^+(q_x)$, $\chi_x^-(q_\Lambda) = \chi^-(q_x)$, $s_x = \operatorname{sign} q_x$. If the following contour bound holds

$$\left\langle \prod_{\langle x, x'\rangle \in \Gamma} \chi_x^+ \chi_{x'}^- \right\rangle_\Lambda \leqslant e^{-E|\Gamma|}, \tag{1.8}$$

where Γ is a set of nearest neighbors, E is independent of Λ and sufficiently large then there exist positive numbers a, a' such that

$$\langle \chi_x^+ \chi_y^- \rangle \leqslant a' e^{-aE}. \tag{1.9}$$

Moreover, if the average is invariant under the change of signs of variables then the ferromagnetic lro occurs in the system, i.e.,

$$\langle s_x s_y \rangle > 0. \tag{1.10}$$

(1.10) follows in a simple way from (1.9) if one takes into account that $\chi_x^{+(-)} = \frac{1}{2}[1 + (-)\, s_x]$. As a result

$$4\langle \chi_x^+ \chi_y^- \rangle_\Lambda = 1 + \langle s_x\rangle_\Lambda - \langle s_y\rangle_\Lambda - \langle s_x s_y\rangle_\Lambda.$$

So, for the systems invariant under the transformation of changing signs of the oscillator variables $\langle s_x\rangle_\Lambda = \langle s_y\rangle_\Lambda = 0$ and the following equality is true

$$\langle s_x s_y\rangle_\Lambda = 1 - 4\langle \chi_x^+ \chi_y^- \rangle_\Lambda.$$

(1.10) holds if the average in the r.h.s.in the equality is strictly less than $\frac{1}{4}$. But this is guaranteed by the large E and (1.9).

The condition of invariance of the average, determined by the correlation functions $\rho^\Lambda(q_\Lambda;t)$, under the change of signs of variables is guaranteed by our choice of U_g, the initial correlation functions and invariance of the Wiener measure under the transformation (the change of a sign of an initial coordinate is equivalent to the change of a sign of a path in the Wiener integral).

We derive the contour bound with the help of the following bound which is a simplified version of a bound from refs. 1 and 2

$$\prod_{\langle x, x' \rangle \in \Gamma} \chi^+(q_x)\, \chi^-(q_{x'}) \leqslant \exp\left\{-\frac{\beta}{2}\, [e_0\, |\Gamma| - Q_{g,\Gamma}(q_\Lambda)]\right\}, \tag{1.11}$$

where e_0 is the minimum for u_g^0, $e_0 = (\eta n)^{-\frac{1}{2(n-1)}} g^{\frac{n}{2(n-1)}}$, and

$$Q_{g,\Gamma}(q_\Lambda) = \sum_{\langle x, x' \rangle \in \Gamma} Q_g(q_x, q_y),$$

$$Q_g(q_x, q_y) = e_0^{-1}[(q_x - q_y)^2 + \tfrac{4}{3}(|q_x^2 - e_0^2| + |q_y^2 - e_0^2|)].$$

The idea to apply the analogue of (1.11) goes back to ref. 10. The presence of the second term in the expression for Q_g hints that one has to control fluctuation of oscillators around the minima of u_g^0 in order to have the lro.

The corner stone of our method is the following non-trivial superstability bound

$$\rho_*^\Lambda(\omega_X) = \rho^\Lambda(\omega_X + e_0) \leqslant \exp\left\{|X|\, E_*(g) - \beta \left[U^+(\omega_X) + \sum_{x \in X} u^+(\omega_x) \right]\right\} \tag{1.12}$$

where U^+ is the part of U generated by the pair ferromagnetic potential, i.e., the first two terms in the expression for φ,

$$\sup_{g \geqslant 1} e_0^{-l} E_*(g) < \infty, \quad 0 < l < 1; \qquad \sup_{g \geqslant 1} (\ln I'_*(g) - \sigma\beta e_0) < \infty, \quad \sigma \ll 1, \tag{1.13}$$

and

$$I'_*(g) = \int e^{-\beta[u^+(\omega) - Q_g^0(\omega)]} P_0(d\omega), \qquad Q_g^0(\omega) = \frac{2}{3e_0}\, |q(q + 2e_0)|,$$

Substituting (1.11) into the left-hand side of (1.8), taking into account the translation invariance of the measure P_0 and (1.12) we conclude that (1.8) holds with

$$E = \frac{\beta}{2}\, e_0 - 2(E_* + \ln I'_*(g)), \qquad e_0(g) > 2. \tag{1.14}$$

From (1.13) and (1.14) it follows that E in Lemma 1.1 can be made arbitrary large by tending g to infinity. Hence, the lro occurs for our systems for sufficiently large g.

The coefficient e_0^{-1} before the translation invariant part of Q_g controls the strength of the pair ferromagnetic n-n interaction in (1.2) as it follows from (1.12). Our main result still holds if the strength, i.e., a coefficient before the term, decreases for growing g not so fast as e_0^{-1} (after the rescaling described above the strength still grows in g).

The paper is organized as follows. The main Theorem 2.1 is formulated in the next section where (1.12) is proven with the help of Theorems 2.2, Corollary 2.1 and Lemma 2.1. Theorem 2.1 is a generalization of the Ruelle superstability bound with properties described in Corollary 2.1 and Lemma 2.1.

Lemma 2.1, Theorem 2.2 and Lemma 1.1 together with (1.11) and Corollary 2.1 are proven in the third, fourth and fifth sections, respectively.

2. MAIN RESULT

We'll use the following notations

$$W(q_{X_1}; q_{X_2}) = U(q_{X_1 \cup X_2}) - U(q_{X_1}) - U(q_{X_2}),$$

$$\|\Psi\|_1 = \sum_x |\Psi(x)|, \qquad |x| = \sup_{v=1,\dots,d} |x^v|,$$

where the summation is performed over $\mathbb{Z}^d$. If U contains an index then the same index will appear in W.

We require that U^0 satisfies the following regularity condition[11, 12]

$$|W^0(q_{X_1}; q_{X_2})| \leqslant \tfrac{1}{2} \sum_{x \in X_1,\, y \in X_2} \Psi^0(|x-y|)[q_x^2 + q_y^2]. \tag{2.1}$$

Theorem 2.1. Let the potential energy of the non-equilibrium system of Brownian linear oscillators with the equation of motion (1.1) be given by (1.2) and $d > 1$. Let, also, the initial distribution of the system be Gibbsian with the potential energy U_0 given by (1.3) and (2.1), where Ψ^0 is a positive function with a finite support independent of g. Then for $\beta > 0$ and for sufficiently large g the ferromagnetic lro occurs for the spins s_x, i.e., $\langle s_x s_y \rangle > 0$.

It is expected that there is a critical value g_c below which there is no lro for the given β and a phase transition occurs. Such the critical value should be calculated approximately from the inequality $a' e^{-aE} \geqslant \frac{1}{4}$. The proof of the phase transition demands a construction of a convergent high-temperature cluster (polymer) expansion which implies an exponential decrease of correlations as in ref. 13. In ref. 14 we wrote down the polymer expansion for quantum oscillator Gibbs systems of linear oscillators with ternary interaction, treating them as Gibbs closed path systems, and proved its convergence at high temperatures. The same result (to be published) is established by the author for the diffusion Gibbs path system (1.5) and (1.6). It can be shown that the condition $\kappa > \frac{1}{2}$ is sufficient for the existence of the thermodynamic limit in the high-temperature phase for the diffusion Gibbs path systems. We do not attribute any physical meaning to it.

To prove Theorem 2.1 we have only to derive (1.12). This will be done with the help of a generalization of the Ruelle superstability bound which is based on application of the superstability and regularity conditions for a potential energy.

We'll deal with the measure space (Ω_*, P_0) that contains subsets B_r, $r > 0$, such that $B_r \subset B_s$ for $s > t$, $P_0(B_r) < \infty$. In the case of the non-equilibrium systems we put $B_r = \{w: |w(t)| \leqslant r\}$. Let $\omega_X \in (\Omega_*)^{|X|}$, $X \subset \mathbb{Z}^d$, $|X| < \infty$. A measurable function $U(\omega_X)$ is required to satisfy the superstability and regularity conditions

$$U(\omega_X) - U^+(\omega_X) \geqslant \sum_{x \in X} u^-(\omega_x), \tag{2.2}$$

$$|W(\omega_{X_1}; \omega_{X_2})| \leqslant \tfrac{1}{2} \sum_{x \in X_1,\, y \in X_2} \Psi(|x-y|)[v(\omega_x) + v(\omega_y)],$$

$$X_1 \cap X_2 = \varnothing, \quad v \geqslant 0 \tag{2.3}$$

where

$$U^+(\omega_X) = \sum_{x,\, y \in X} \varphi^+_{x,\, y}(\omega_x, \omega_y), \qquad \varphi^+ \geqslant 0,$$

and all the functions are measurable. The following integrals are necessary attributes of the superstability bound since its main constant c_0 depends on them

$$\bar{u}(\omega) = U(\omega) + \|\Psi\|_1\, v(\omega), \quad I_r = e^{-\frac{1}{2}\beta \|\Psi\|_1 \bar{v}_r} I_0, \qquad I_0 = \int_{B_r} e^{-\beta \bar{u}(\omega)} P_0(d\omega),$$

$$I(\varepsilon) = \int \exp\{-\beta[u^-(\omega) - 3\varepsilon v(\omega)]\}\, P_0(d\omega), \qquad \bar{v}_r = \operatorname*{ess\,sup}_{\omega \in B_r} v(\omega).$$

Theorem 2.2. Let $\rho^{\Lambda}(\omega_X)$ be given by the second equality in (1.5), the function U satisfy (2.2) and (2.3). Let's put $\psi(x)=|x|^k$, $l_j=(1+2\alpha)^j$ and require that

$$\|\psi\Psi\|_1 \leqslant \infty, \qquad \|\Psi\|_1\,[(1+3\alpha)^{2(d+k)}-1] \leqslant \frac{\varepsilon}{2},$$

where $0<3\varepsilon<1$. Then the following superstability bound is valid

$$\rho^{\Lambda}(\omega_X) \leqslant \exp\left\{-\beta\left[U^+(\omega_\Lambda)+\sum_{x\in X}(u^-(\omega_x)-3\varepsilon v(\omega_x))\right]+|X|\,c_0(\varepsilon, I_r^{-1}, I(\varepsilon))\right\}, \tag{2.4}$$

and there exist the positive numbers c^0, ξ such that for $V_j=(1+2l_j)^d$ the following representation is true

$$c_0(\varepsilon, z', z)=c^0+\ln(1+\xi z'+f(\varepsilon, zz')), \qquad f(\varepsilon, z)=\sum_{j\geqslant 0} e^{-\varepsilon\psi(l_j)V_j}(2z)^{V_j}. \tag{2.5}$$

Moreover, if Ψ has a finite support then c^0 does not depend on ε and ξ is bounded in ε.

The bound (2.4) differs from the Ruelle bound by the presence of $e^{-\beta U^+}$, more general ψ and the condition for α ($k=1$ in the Ruelle condition) and, also, less general l_j (see the beginning of the fourth section).

c^0 is determined by the equality in (4.12) and its dependence on ε is given by the conditions of Lemma 4.1.

The exponential term appears in the bounds of the correlation functions of classical oscillator systems in the high-temperature phase if one applies the technique of the Kirkwood-Saltsburg relations proposed in ref. 13. The similar term is present in the superstability bound in ref. 15 for classical oscillator ferromagnetic systems with a special pair potential.

Our ψ enables to control an asymptotics of c_0 in ε needed for providing the right asymptotics of $E_*(g)$ in g after putting $\varepsilon=e_0^{-2}$. It turns out that c_0 grows very slowly for the vanishing ε for fixed z', z in (2.5) only for short range interactions. The character of the asymptotics is depicted in the following corollary.

Corollary 2.1. Let the conditions of Theorem 2.2 be satisfied and $k\geqslant d$

(a) then the following inequality is true

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{2d}{k+d}} \ln f(\varepsilon, z) \leqslant 2^{d\frac{2k+d}{k+d}}(\ln 2z)^2$$

(b) Moreover, if Ψ has a finite support then the following inequality holds

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{2d}{k+d}} c_0(\varepsilon, z, z') \leqslant 2^{d\frac{2k+d}{k+d}} (\ln 2zz')^2.$$

Since the measure P_0 is translation invariant the correlation functions ρ_*^A are expressed in terms of the potential energy U_*

$$U_*(\omega_X) = U(\omega_X + e_0) - |X|\, \kappa u_g^0(e_0).$$

The significant fact is that U_* has a limit when g tends to infinity (see (3.5) and (3.6)).

It is not difficult to prove (2.2) and (2.3) for U_* and to calculate all the functions in these equalities. It is done in the next section. If Ψ^0 has a finite support that the same is true for Ψ.

The next lemma reduces a proof of (1.12) to a determination of the asymptotics of c_0 in $\varepsilon = e_0^{-2}$ for fixed z, z' in (2.5) and allows to apply Theorem 2.2 and Corollary 2.1 for proving Theorem 2.1 via formulas (1.12)–(1.14).

Lemma 2.1. Let u^-, v be the functions in the superstability and regularity conditions (2.2) and (2.3) for U_* and $\bar{u}_*(\omega) = U_*(\omega) + \|\Psi\|_1\, v(\omega)$. Let, also, the integrals $I(e_0^{-2}), I_r(e_0^{-2})$, associated to U_*, be denoted by $I_*(g), I_{*r}(g)$, respectively, and

$$u^+(\omega) = u^-(\omega) - 3e_0^{-2} v(\omega).$$

Then the functions $(I_{*r})^{-1}(g)$, $I_*(g)$, $e^{-\sigma\beta e_0} I'_*(g)$, where σ is an arbitrary small number, are bounded in g. For short-range interaction k may be arbitrary. Thus, Lemma 2.1 and Corollary 2.1 give the following final result.

Corollary 2.2. Let the conditions of Theorem 2.1 be satisfied, c_0, $k > 3d$ and u^+ be determined in Theorem 2.2 and Lemma 2.1, respectively. Let's put

$$E_*(g) = c_0(e_0^{-2}, I_{*r}(g), I_*(g)), \tag{2.6}$$

then the inequalities in (1.12) and (1.13) are true with $l = \frac{4d}{k+d}$.

This corollary, (1.12)–(1.14) and arguments from the previous section prove Theorem 2.1

The most important properties of the functions u_*, u^-, v, Q_g^0 that allow to prove Lemma 2.1 are:

(a) the first function converges to a sum of functions $q^2, w^2(t)$, $\int_0^t w^2(\tau)\, d\tau$ with positive coefficients when g tends to infinity;

(b) u^- differs from u_* only by a constant independent of g;

(c) $e_0^{-2}v$ is bounded for $|w(t)| = 2e_0$ and $Q_g^0(q)$ is bounded in a neighborhood of $-2e_0$.

Item (b) is a consequence of the superstability bound (3.1) for $V(q_X)$. Without it one can only prove an existence of the lro on the time interval $[0, e_0^{-2}]$.

We estimate the integrals I_*, I'_* with the help of the inequality proven in the next section

$$\int dq\, F_0(q) \int P_q(dw) \exp\left\{\int_0^t v(w(\tau))\, d\tau\right\} F^0(w(t)) \leqslant \|e^{tv}\|_\infty\, \|F^0\|_\infty\, \|F_0\|_1, \tag{2.7}$$

where $\|\cdot\|_\infty$ is the norm of $\L^\infty$, i.e., the space of bounded functions.

Remark 2.1. Writing this paper we found out a flaw in arguments in refs. 1 and 2: one of them is based on the assumption of boundedness of the integral

$$\int e^{-\beta[u_g^0(q+e_0)-u_g^0(e_0)]} e^{\alpha e_0^{-2s} q^{2l}}\, dq, \qquad \alpha > 0, \quad 0 < l < n,$$

where the integration is performed over $\mathbb{R}$, for $s < l$. But the integral is not bounded in $g \geqslant 1$ since the second exponent is not bounded in g at infinity in the neighborhood of the critical point $-2e_0$ (the Lebesque dominated convergence theorem is not applicable in spite of the fact that the function under the sign of the integral converges to the integrable function in the limit of the infinite g). The proof of Lemma 2.1. is based on boundedness of the integral for $s \geqslant l$ which is possible only after application of the new version of the Ruelle superstability bound (2.4). The analogs of Theorem 2.2 and Corollary 2.1 hold also for classical and quantum oscillator systems considered in refs. 1 and 2. The mentioned flaw can be eliminated with their use without difficulty only for finite-range interactions.

The obstruction for the generalization of Theorem 2.1 and its analogs for the Gibbs classical and quantum linear oscillator systems for infinite-range interactions is the dependence of c^0, ξ on ε (c^0 is determined by the first term in (4.12)). The first two parameters grow faster then e_0 if $\varepsilon = e_0^{-2}$. This obstruction may be overcome only if a generalization of the Ruelle arguments, which yields a weaker growth of c^0, ξ in ε at zero, is found.

3. INEQUALITIES FOR U_* AND LEMMA 2.1

In this section we obtain expresions for the functions u^-, v in (2.2) and (2.3) for U_* and prove Lemma 2.1.

The derivation of the superstability bound (2.2) relies on the superstability bound for the function V

$$V(q_X) \geqslant \sum_{x \in X} V(q_x) - 2(n-1)\,|X|, \tag{3.1}$$

where $V(q_x)$ is calculated from the expression for $V(q_X)$ using the equality $U_g(q_x) = u_g^0(q_x)$.

This superstability condition for V is proven by induction. Indeed, let $x \notin X$ and X_x be the set of the n-n of x in X and $|q_x| \geqslant |q_X|$. Then

$$\begin{aligned} V(q_{x \cup X}) \geqslant{} & V(q_X) - \beta \sum_{y \in X_x} (q_x - q_y)^2 \\ & - \frac{1}{2}\,\partial^2 u_g^0(q_x) + \frac{\beta}{4}\left[\partial u_g^0(q_x) + 2 \sum_{y \in X_x} (q_x - q_y)\right]^2. \end{aligned}$$

If $q_x \geqslant e_0$ then $\partial u_g^0(q_x) \geqslant 0$ (see (3.6)) and

$$\left[\partial u_g^0(q_x) + 2 \sum_{y \in X_x} (q_x - q_y)\right]^2 \geqslant [\partial u_g^0(q_x)]^2 + 4\left[\sum_{y \in X_x} (q_x - q_y)\right]^2$$

Hence, (3.1) holds even without the last term in its right-hand side.

If $q_x \leqslant e_0$ then (see (3.8))

$$\begin{aligned} V(q_{X \cup x}) &\geqslant -\tfrac{1}{2} \sum_{y \in X \cup x} \partial^2 u_g^0(q_y) \\ &\geqslant -\tfrac{1}{2} \sum_{y \in X \cup x} \partial^2 u_g^0(e_0) = -2(n-1)(|X|+1), \qquad |q_{X \cup x}| \leqslant e_0. \end{aligned}$$

The case $q_x \leqslant 0$ is mapped into the case $q_x \geqslant 0$ by changing signs of all variables.

As a result

$$V(q_X + e_0) \geqslant \sum_{x \in X} V(q_x + e_0) - 2(n-1)\,|X|. \tag{3.2}$$

From (3.2) we derive the superstability bound (2.2) for U_* with

$$u^-(\omega) = u_*(\omega) - 2(n-1)\, t. \tag{3.3}$$

Now, we have to estimate at first the potential ϕ determined in the first section. From the definition of u_g^0 we get

$$u_g^0(q) = n^{-1}(e_0^{2-2n} q^{2n} - nq^2), \tag{3.4}$$

$$u_g^0(q+e_0) - u_g^0(e_0) = 2(n-1)\, q^2 + r_0(q),\ \ r_0(q) = e_0^2 n^{-1} \sum_{j=3}^{2n} \frac{j!(2n-j)!}{n!} (e_0^{-1} q)^j. \tag{3.5}$$

From (3.4) we obtain

$$\begin{aligned} \partial u_g^0(q+e_0) &= 2(e_0^{2-2n}(q+e_0)^{2n-1} - (q+e_0)) = 2(2(n-1)\, q + r_1(q)), \\ r_1(q) &= e_0 \sum_{j=2}^{2n-1} C_{2n-1}^j (e_0^{-1} q)^j. \end{aligned} \tag{3.6}$$

As a result

$$\begin{aligned} \phi(q_x+e_0, q_y+e_0) &= (\partial u_g^0(q_x+e_0) - \partial u_g^0(q_y+e_0))(q_x - q_y) \\ &= 2(r_1(q_x) - r_1(q_y)(q_x - q_y) + 4(n-1)(q_x - q_y)^2. \end{aligned}$$

Hence

$$\begin{aligned} &|\phi(q_x+e_0, q_y+e_0)| \\ &\quad \leqslant 2[(|q_x|\, r_1(|q_x|) + |q_y|\, r_1(|q_y|)) + (|q_y|\, r_1(|q_x|) + |q_x|\, r_1(|q_y|)] \\ &\quad\quad + 8(n-1)(q_x^2 + q_y^2). \end{aligned}$$

From the inequality

$$a^j c + c^j a \leqslant (a+c)^{j+1} \leqslant 2^{j+1}(a^{j+1} + c^{j+1}), \qquad a, c \geqslant 0,$$

we derive

$$\begin{aligned} |\phi(q_x+e_0, q_y+e_0)| &\leqslant [v^-(q_x) + v^-(q_y)], \\ v^-(q) &= 8e_0^2 \sum_{j=2}^{2n-1} 2^{j+1} C_{2n-1}^j\, |e_0^{-1} q|^{j+1} + 8(n-1)\, q^2. \end{aligned} \tag{3.7}$$

It is easy to check that the contribution of the pair potential to the right-hand side of (2.3) for U_* is expressed through the function

$$v^-(w)+q^2+2(\kappa-\tfrac{1}{2})\, w^2(t), \qquad v^-(w)=\beta\int_0^t v^-(w(\tau))\, d\tau.$$

We have to estimate the contribution to (2.3) of U'_*. Let $\delta_{Y_3}=\delta_{|y_1-y_2|,\,1}\delta_{|y_1-y_3|,\,1}$ then

$$\begin{aligned}|W'(q_{X_1}+e_0;\, q_{X_2+e_0})| &\leqslant 2^5\sum_{y_1\in X_1}\ \sum_{y_2,\, y_3\in X_2}\delta_{Y_3}(q_{y_1}^2+q_{y_2}^2+q_{y_3}^2)\\ &\leqslant 2^6\sum_{y_1\in X_1}\ \sum_{y_2\in X_2}(q_{y_1}^2+q_{y_2}^2)\sum_{y_3}\delta_{Y_3}\\ &\leqslant 2^7 d\sum_{y_1\in X_1}\ \sum_{y_2\in X_2}\delta_{|y_1-y_2|,\,1}(q_{y_1}^2+q_{y_2}^2).\end{aligned}$$

Hence, from the last inequalities it follows that (2.3) holds for U_* with

$$\Psi(|x|)=\Psi^0(|x|)+2\delta_{|x|,\,1},$$

$$v(\omega)=v^-(w)+q^2+(2\kappa-1)\, w^2(t)+2^7 d\beta\int_0^t w^2(\tau)\, d\tau.$$

Proof of Lemma 2.1. For the second derivative of the potential u_g^0 we have the following equalities

$$\begin{aligned}\partial^2 u_g^0(q)&=2((2n-1)\, e_0^{2-2n}q^{2n-2}-1),\\ \partial^2 u_g^0(q+e_0)&=2((2n-1)\, e_0^{2-2n}(q+e_0)^{2n-2}-1)\\ &=4(n-1)+2r_2(q), \quad r_2(q)=\sum_{j=1}^{2n-2} C_{2n-2}^j(e_0^{-1}q)^j.\end{aligned} \tag{3.8}$$

Since polynomials r_s, $s=0, 1, 2$ in (3.5), (3.6), and (3.8) tend to zero in the limit of infinite g the functions u_*, $\bar{u}_*(\omega)$, v converge to quadratic functions in q^2, w in the limit.

The function $\bar{u}_*(\omega)$ is uniformly bounded in $g\in[1, \infty]$ in B_r

$$|\bar{u}_*(\omega)|\leqslant \bar{Q}(r), \qquad \omega\in B_r,$$

where $\bar{Q}(r)$ is a polynomial in r depending, also, on t. This inequality is easily obtained taking into account that r_s, $s=0, 1, 2$ are positive polynomials in $|q|$ and e_0^{-1}. As a result

$$\sup_{g\geqslant 1} I_{*r}^{-1}\leqslant e^{\beta\bar{Q}(r)}\left(\int_{B_r} P_0(d\omega)\right)^{-1}<\infty. \tag{3.9}$$

From the definition of $u_*(\omega)$ we get

$$u^+(\omega) = u_0^+(q) + u_1^+(w(t)) + \int_0^t u_2^+(w(\tau))\, d\tau,$$

$$u_0^+(q) = [u_g^0(q+e_0) - u_g^0(e_0)] - 3e_0^{-2} q^2,$$

$$u_1^+(q) = \left(\kappa - \frac{1}{2}\right)[u_g^0(q+e_0) - u_g^0(e_0)] - 3e_0^{-2}(2\kappa - 1)\, q^2,$$

$$u_2^+(q) = \left[-\frac{1}{2}\,\partial^2 u_g^0(q+e_0) + \frac{\beta}{4}\,(\partial u_g^0)^2(q+e_0)\right] - e_0^{-2}\beta(2^7 q^2 d + v^-(q)) - 2(n-1).$$

Then (2.7) yields

$$I_*(g) \leqslant \|e^{-t\beta u_2^+}\|_\infty\, \|e^{-\beta u_1^+}\|_\infty\, \|e^{-\beta u_0^+}\|_1, \tag{3.10}$$

$$I'_*(g) \leqslant \|e^{-t\beta u_2^+}\|_\infty\, \|e^{-\beta u_1^+}\|_\infty\, \|e^{-\beta(u_0^+ - Q_g^0)}\|_1. \tag{3.11}$$

A real line is represented as a union of three sets:

$$q \geqslant 0, \qquad q \leqslant -2e_0, \qquad |q| \leqslant 2e_0.$$

Then every norm in the last inequalities is less than the sum of the norms with the functions restricted to these three sets. In the first set the functions given by (3.5) and (3.6) are positive and all the coefficients before q^j, $j > 1$ in their expressions are greater than the corresponding coefficients in the rest of the function in the expressions of u_s^+, hence, for sufficiently large g the functions u_s^+ are bounded from below by a positive quadratic polynomial. In Q_g^0 the term linear in $|q|$ does not depend on g and is majorized by this polynomial. As a result the three norms are bounded in g in the first set. The second set is transformed into the first by translating the variables by $-2e_0$ and changing their signs. This transformation does not change the functions in the square brackets in the expressions for u_s^+, so, the three norms, also, are bounded in the second set. The first two norms are bounded in g in the third set since all the functions are either positive or depend on $\frac{|q|}{e_0}$. Hence the first two norms are bounded in g on a whole line.

Now, we have to deal with the third norm in (3.10) and (3.11), i.e., the corresponding integrals.

In the third set we rescale the variable in it by e_0. Then

$$u_g^0(e_0 q + e_0) - u_g^0(e_0) = e_0^2 h(q), \quad h(q) = n^{-1}[n - 1 + (q+1)^{2n} - n(q+1)^2],$$

$$Q_g^0(e_0 q) = e_0\, \tfrac{2}{3}\, |q|\, |q+2|.$$

The third set is mapped into interval $[-2.2]$ and the two integrals over this set is multiplied by e_0. The last function in the expression for the rescaled u_0^+ is bounded in the set in q and g. The non-negative function h has two zeros at the points $y_1 = 0$, $y_2 = -2$. In a sufficiently small corresponding neighborhoods (intervals) $Y_s = \{q \colon |q - y_s| \leqslant \frac{\sigma'}{2}\}$, $\sigma' \ll 1$, $s = 1, 2$, of the points it is bounded from below by a positive quadratic polynomial $\alpha(q - y_s)^2$ since all the coefficients before $(q - y_s)^2$ in other terms in its Taylor expansions are less then $\frac{\sigma'}{2} \ll 1$. In the intervals Y_s we have $Q_g^0(e_0 q) \leqslant \sigma e_0$, where $\sigma = \frac{3}{4}\sigma'$. So, the integral in Y_s, corresponding to the third norms in (3.10) and the integral in Y_s, corresponding to the third norm in (3.11), multiplied by $\exp\{-\beta\sigma e_0\}$ are less than

$$e_0 C \int_{q \in Y_s} e^{-e_0^2 \alpha\beta(q - y_s)^2} \leqslant e_0 C \int e^{-e_0^2 \alpha\beta(q - y_s)^2}\, dq = C \int e^{-\alpha\beta q^2}\, dq, \qquad s = 1, 2,$$

where $C = e^{\frac{4}{3}}$ corresponds to the contribution of the last term in the expression for u_0^+.

In the complement to Y_s the third norms are less than $2Ce_0 \exp\{-\alpha' e_0^2 + \frac{16\beta}{3} e_0\} \leqslant C'$. Hence, the third norm in (3.10) is bounded in g and the third norm in (3.11) is bounded by

$$C e^{\beta\sigma e_0} \int e^{-\alpha\beta q^2}\, dq + C'.$$

Since σ is arbitrary small we conclude that Lemma 2.1 is true.

Proof of (2.7). From the Feynman–Kac and Lie–Trotter formulas we derive

$$I_t(F \mid v) = \lim_{n \to \infty} I_t(F \mid v, n),$$

$$I_t(F \mid v, n) = \int dq\, F_0(q) \int dq_1 \cdots dq_n\, p_0^{tn^{-1}}(q - q_1)\, e^{tn^{-1} v(q_1)}$$

$$\times \prod_{j=2}^{n} p_0^{tn^{-1}}(q_j - q_{j-1})\, e^{tn^{-1} v(q_j)} F^0(q_n),$$

where

$$p_0^t(q) = (\sqrt{4\pi t})^{-1}\, e^{-\frac{q^2}{4t}}.$$

Now, lets apply the generalized Holder inequality for a measure μ on arbitrary measure space (the usual Holder inequality and induction are used)

$$\int \prod_{j=1}^{n} f_j \, d\mu \leqslant \prod_{j=1}^{n} \left[\int |f_j|^n \, d\mu \right]^{\frac{1}{n}},$$

taking $\mathbb{R}^{n+1}$ as the measure space and putting $f_j(q, q_1,..., q_n) = e^{\frac{t}{n} v(q_j)}$,

$$d\mu(q, q_1,..., q_n)$$

$$= |F_0(q)| \, dq \, dq_1 \cdots dq_n \, p_0^{tn^{-1}}(q-q_1) \prod_{j=2}^{n} p_0^{tn^{-1}}(q_j - q_{j-1}) \, |F^0(q_n)|.$$

From the semigroup property of p_0^t it follows that

$$I_t(F \mid v, n)$$

$$\leqslant \prod_{j=1}^{n-1} \left[\int dq \, |F_0(q)| \int dq_1 \, dq_2 \, p_0^{t\frac{j}{n}}(q-q_1) \, e^{tv(q_1)} \, p_0^{t\frac{n-j}{n}}(q_1 - q_2) \, |F^0(q_2)| \right]^{\frac{1}{n}}$$

$$\times \left[\int dq \, |F_0(q)| \int p_0^t(q-q') \, e^{tv(q')} \, |F^0(q')| \, dq' \right]^{\frac{1}{n}}$$

$$\leqslant \|e^{tv}\|_\infty \int dq \, dq' \, |F_0(q)| \, p_0^t(q-q') \, |F^0(q')|$$

$$\leqslant \|e^{tv}\|_\infty \, \|F^0\|_\infty \int dq \, |F_0(q)| \int p_0^t(q-q') \, dq'$$

This concludes the proof of (2.7).

4. SUPERSTABILITY BOUND AND ITS PROPERTIES

In this section we prove Theorems 2.2 and 2.3. We start from Theorem 2.2. Let's put as in ref. 11

$$[j] = \{x \in \mathbb{Z}^d : |x| = \max_s |x^s| \leqslant l_j\}, \qquad \psi_j = \psi(l_j).$$

There are constants A, B and the integer P_0 in ref. 11. We put $A = 2\varepsilon$, $B = 0$, $P_0 = 0$ here. The last condition is a consequence of our choice of the sequence l_j which satisfies in ref. 16 the more general condition

$$\left|\frac{l_{j+1}}{l_j}-(1+2\alpha)\right|<\alpha, \qquad j\geqslant P_0.$$

We'll assume that instead of (2.2) the following condition holds

$$U_-(\omega_X)=U(\omega_X)-U^+(\omega_X)\geqslant \sum_{x\in X} v(\omega_x), \tag{4.1}$$

and we'll prove the superstability bound

$$\rho^\Lambda(\omega_X)\leqslant \exp\left\{-\beta\left[U^+(\omega_X)+\sum_{x\in X}(1-3\varepsilon)\,v(\omega_x)\right]+c_0\,|X|\right\}.$$

(4.1) implies the following change: the new potential energy, measure P_0, and correlation functions are connected with the old corresponding quantities by subtracting the sum

$$\sum_{x\in\Lambda}[u^-(\omega_x)-v(\omega_x)],$$

multiplying by $\exp\{-\beta[u^-(\omega)-v(\omega)]\}$ and $\exp\{\sum_{x\in\Lambda}[u^-(\omega_x)-v(\omega_x)]\}$. respectively.

Let's put $n^2(x)=v(\omega_x)$,

$$R_P^0=\left\{\omega_\Lambda\colon \sum_{x\in[s]} n^2(x)\leqslant\psi_s V_s,\ s\geqslant P\right\},$$

$$R_q=\left\{\omega_\Lambda\colon \sum_{x\in[q]} n^2(x)\geqslant\psi_q V_q,\ \sum_{x\in[q+1]} n^2(x)\leqslant\psi_{q+1}V_{q+1}\right\}$$

$$\tilde\rho^\Lambda(\omega_\Lambda)=e^{\beta U^+(\omega_\Lambda)}\rho^\Lambda(\omega_\Lambda).$$

Superstabilty bound (2.4) is derived from the following two bounds which coincide with the bounds (2), (4) from ref. 16. This coincidence is the most important fact for our proof

$$\tilde\rho'(\omega_X)=\tilde\rho^\Lambda(\omega_X\mid R_P^0)\leqslant C'e^{-\beta(1-\|\Psi\|_1)\,v(\omega_x)}\tilde\rho^\Lambda(\omega_{X\backslash x}), \tag{4.2}$$

$$\tilde\rho''(\omega_X)=\tilde\rho^\Lambda(\omega_X\mid(\Omega_*)^\Lambda\backslash R_P^0)\leqslant\sum_{s\geqslant P} e^{-(\beta\varepsilon\psi_{s+1}V_{s+1}-D''V_{s+1})}$$
$$\times\exp\left\{-\beta\sum_{x\in[s+1]\cap X}(1-3\varepsilon)\,v(\omega_x)\right\}\tilde\rho^\Lambda(\omega_{X\backslash[s+1]}),\quad \tilde\rho^\Lambda(\varnothing)=1, \tag{4.3}$$

where $\tilde{\rho}'$, $\tilde{\rho}''$ is obtained by inserting $\chi_{R_P^0}$ (characteristic function of R_P^0), $1-\chi_{R_P^0}$, respectively, into the integral defining $\tilde{\rho}^{\Lambda}$ and

$$C' = e^{\beta(D' + \frac{1}{2}\|\Psi\|_1 \bar{v}_r)} \left(\int_{B_r} e^{-\beta \bar{u}(\omega)} P_0(d\omega) \right)^{-1},$$

$$e^{\beta D''} = C' e^{-\beta D'} \left(1 + \int e^{\beta(1-3\varepsilon)\, v(\omega)} P_0(d\omega) \right).$$

The constant D' is determined in Proposition 4.1.

The proof of (4.2) and (4.3) relies on the following lemma and proposition in which the number P depends on ε only for Ψ with an infinite support (Lemma 2.4 from ref. 11) since $l_{P+k} - l_P = (1+2\alpha)^P [(1+2\alpha)^k - 1]$, $k \geqslant 1$, tends to infinity for growing P. If interaction has a finite support then P depends on its radius.

Lemma 4.1. Let the conditions of Theorem 2.2 be satisfied and the number P is such that

$$\sum_{|x| \geqslant l_{P+1} - l_P} \Psi(|x|) \leqslant 2^{-1}\varepsilon,$$

$$\sum_{k \geqslant 1} [\Psi(l_{P+k+1} - l_{P+1}) - \Psi(l_{P+k+2} - l_{P+1})]\, \psi_{k+P+2} V_{k+P+2} \leqslant 2^{-1}\varepsilon,$$

$$\psi_P \geqslant (1+3\alpha)^d.$$

Then for $q \geqslant P$ and $\omega_\Lambda \in R_q$ the following inequality holds

$$\sum_{x \in [q+1],\, y \notin [q+1]} \Psi(|y-x|)[n^2(x) + n^2(y)] \leqslant \varepsilon \sum_{x \in [q+1]} n^2(x). \tag{4.4}$$

For finite-range interactions the conditions of Lemma 4.1 are obvious for arbitrary small ε. For infinite-range interactions they have to be proved for such the ε (see Lemma 2.4 in ref. 16).

Proposition 4.1. For $\omega_\Lambda \in R_P^0$ the following inequality is true

$$\sum_{x \notin [P]} \Psi(|x|)\, n^2(x) \leqslant D' - \psi_P V_P \Psi(0),$$

where

$$D' = \Psi(0)\, \psi_P V_P + (1+3\alpha)^{d+k} \sum_{l \geqslant l_P} [\Psi(l) - \Psi(l+1)]\, \psi(l)(l+1)^d.$$

It's clear that the last condition in Lemma 4.1 is satisfied for $P \geqslant 2\, dk^{-1}$. The proof of (4.2) and (4.3) relies, also, on the inequalities which are not used in ref. 11

$$\begin{aligned} U(\omega_\Lambda) &- U^+(\omega_X) + U(\omega'_x) \\ &\geqslant U_-(\omega_x) + [U(\omega'_x, \omega_{\Lambda\setminus x}) - U^+(\omega_{X\setminus x})] \\ &\quad + W_-(\omega_x; \omega_{\Lambda\setminus x}) - W(\omega'_x; \omega_{\Lambda\setminus x}), \end{aligned} \tag{4.5}$$

$$\begin{aligned} U(\omega_\Lambda) &- U^+(\omega_X) + U(\omega'_{[s+1]}) \\ &\geqslant U_-(\omega_{[s+1]}) + [U(\omega'_{[s+1]}, \omega_{\Lambda\setminus[s+1]}) - U^+(\omega_{X\setminus[s+1]})] \\ &\quad + W_-(\omega_{[s+1]}; \omega_{\Lambda\setminus[s+1]}) - W(\omega'_{[s+1]}; \omega_{\Lambda\setminus[s+1]}). \end{aligned} \tag{4.6}$$

These inequalities will be proved in the end of the section.

For the proof of (4.2) one has to put $x = 0$ and to estimate the potential energy $W(\omega_x; \omega_{\Lambda\setminus x_1})$ on R_P^0 with the help of Proposition 4.1 (Ψ is decreasing)

$$\begin{aligned} 2W(\omega_x; \omega_{\Lambda\setminus x}) &\leqslant \sum_y \Psi(|y|)[v(\omega_x) + v(\omega_y)] \\ &\leqslant \|\Psi\|_1\, v(\omega_x) + \Psi(0) \sum_{y \in [P]} v(\omega_y) + \sum_{y \notin [P]} \Psi(|y|)\, v(\omega_y) \\ &\leqslant \|\Psi\|_1\, v(\omega_x) + D' \end{aligned} \tag{4.7}$$

Then one has to separate ω_x from the rest of variables, introduce a new variable $\omega'_x \in B_r$, add and subtract $W(\omega_x; \omega_{X\setminus x})$. In order to obtain (4.2) it is necessary to integrate by $P_0(\omega'_x)$, also, over the set B_r and apply the inequality derived from the regularity condition

$$\int_{B_r} e^{-\beta U(\omega_X)} P_0(d\omega_X) \geqslant \left[\int_{B_r} e^{-\beta \bar{u}(\omega)} P_0(d\omega) \right]^{|X|} \tag{4.8}$$

The applied inequality is given by

$$U(\omega_X) \leqslant \sum_{x \in X} \bar{u}(\omega_x). \tag{4.9}$$

The inequality is derived after applying $|X|-1$ times the equalities, following from the regularity condition (2.3)

$$U(\omega_x, \omega_X) \leqslant U(\omega_x) + U(\omega_X) + |W(\omega_x; \omega_X)|$$

$$\leqslant U(\omega_x) + U(\omega_X) + \|\Psi\|_1\, v(\omega_x) + \sum_{y \in X} \Psi_{|x-y|} v(\omega_y),$$

$$\sum_{x \in X} \sum_{y \in X' \subseteq X} \Psi_{|x-y|} v(\omega_y) \leqslant \|\Psi\|_1 \sum_{y \in X'} v(\omega_y).$$

From (4.8) it follows that

$$\tilde{\rho}'(\omega_X) \leqslant e^{-\beta(D' + \frac{1}{2}\|\Psi\|_1\, \bar{v}_r)} C' Z_\Lambda^{-1} \int_{B_r} e^{-\beta U(\omega'_x)} P_0(d\omega'_x)$$

$$\times \int_{R_P^0} \exp\{-\beta[U(\omega_\Lambda) - U^+(\omega_X)]\}\, P_0(d\omega_{\Lambda \setminus X})$$

From Proposition 4.1, Eq. (4.7) for $W_-(\omega_x; \omega_{\Lambda\setminus x})$ and $W(\omega'_x; \omega_{\Lambda\setminus x})$, (4.4) and the inequality $U_-(\omega) \geqslant v(\omega)$ we obtain, taking the maximum of $v(\omega'_x)$ in B_r

$$\tilde{\rho}'(\omega_X) \leqslant C' e^{-\beta(1-\|\Psi\|_1)\, v(\omega_x)} Z_\Lambda^{-1} \int_{B_r} P_0(d\omega'_x)$$

$$\times \int_{R_P^0} \exp\{-\beta[U(\omega'_x, \omega_{\Lambda\setminus x}) - U^+(\omega_{X\setminus x})]\}\, P_0(d\omega_{\Lambda\setminus X}).$$

Enlarging the domain of integration to $\mathbb{R}^{|\Lambda \setminus X|+1}$ we obtain (4.2).

A proof of (4.3) is based on an application of Lemma 4.1. But, at first, we have to introduce new variables $\omega'_{[s+1]}$, $\omega'_x \in B_r$.

From Lemma 4.1, (4.6), (4.8) and the inequality $U_-(\omega_{[s+1]}) \geqslant \sum_{x \in [s+1]} v(\omega_x)$ we derive ($[s+1] \subset \Lambda$)

$$\tilde{\rho}''(\omega_X)$$

$$\leqslant Z_\Lambda^{-1} \sum_{s \geqslant P} (C' e^{-\beta D' - \beta \frac{1}{2}\|\Psi\|_1\, \bar{v}_r})^{|[s+1]|} \int_{B_r^{|[s+1]|}} P_0(d\omega'_{[s+1]}) \exp\{-\beta U(\omega'_{[s+1]})\}$$

$$\times \int_{R_s} \exp\{-\beta[U(\omega_\Lambda) - U^+(\omega_X)]\}\, P_0(d\omega_{\Lambda\setminus X})$$

$$
\begin{aligned}
&\leqslant Z_\Lambda^{-1} \sum_{s \geqslant P} \left(C' e^{-\beta D' - \beta \frac{\beta}{2} \|\Psi\|_1 \bar{v}_r}\right)^{|[s+1]|} \exp\left\{-\beta \sum_{x \in [s+1] \cap X} v(\omega_x)\right\} \\
&\times \int_{B_r^{|[s+1]|}} P_0(d\omega'_{[s+1]}) \int \exp\{-\beta[U(\omega'_{[s+1]}, \omega_{\Lambda\setminus[s+1]}) - U^+(\omega_{X\setminus[s+1]})]\} \\
&\times \exp\left\{\frac{\beta}{2} \sum_{x \in [s+1],\, y \notin [s+1]} \Psi(|x-y|)[v(\omega_x) + v(\omega_y)]\right\} \\
&\times \exp\left\{\frac{\beta}{2} \sum_{x \in [s+1],\, y \notin [s+1]} \Psi(|x-y|)[v(\omega'_x) + v(\omega_y)]\right\} P_0(d\omega_{\Lambda\setminus[s+1]\setminus X}) \\
&\times \left(\int e^{-\beta(1-3\varepsilon)\, v_g(\omega)} P_0(d\omega)\right)^{|[s+1]\setminus X|}.
\end{aligned}
$$

Now, the following inequalities can be applied

$$
\begin{aligned}
&\exp\left\{\frac{\beta}{2} \sum_{x \in [s+1],\, y \notin [s+1]} \Psi(|x-y|)[v(\omega'_x) + v(\omega_y)]\right\} \\
&\quad \leqslant \exp\left\{\frac{\beta}{2} \sum_{x \in [s+1],\, y \notin [s+1]} \Psi(|x-y|)[v(\omega_x) + v(\omega_y)]\right\} \\
&\qquad \times \exp\left\{\frac{\beta}{2} \sum_{x \in [s+1]} \Psi(|x-y|)\, v(\omega'_x)\right\} \\
&\quad \leqslant \exp\left\{\frac{\beta}{2} \sum_{x \in [s+1],\, y \notin [s+1]} \Psi(|x-y|)[v(\omega_x) + v(\omega_y)]\right\} \\
&\qquad \times \exp\left\{\frac{\beta}{2} |[s+1]|\, \|\Psi\|_1\, \bar{v}_r\right\}.
\end{aligned}
$$

As a result

$$
\begin{aligned}
\tilde{\rho}''(\omega_X) &\leqslant Z_\Lambda^{-1} \sum_{s \geqslant P} e^{\beta\,|[s+1]|\,D''} \exp\left\{-\beta\left[\sum_{x \in [s+1]} (1-3\varepsilon)\, v(\omega_x) + \varepsilon \psi_{s+1} V_{[s+1]}\right]\right\} \\
&\quad \times \int P_0(d\omega'_{[s+1]}) \int \exp\{-\beta[U(\omega'_{[s+1]}, \omega_{\Lambda\setminus[s+1]}) - U^+(\omega_{X\setminus[s+1]})]\} \\
&\quad \times P_0(d\omega_{\Lambda\setminus[s+1]\setminus X}).
\end{aligned}
$$

Here we added and subtracted the term $\varepsilon \sum_{x \in [s+1]} v(\omega_x)$ under the sigh of the exponent. Here the set $\Lambda\setminus[s+1]\setminus X$ coincides with $\Lambda\setminus X$ for $[s+1] \subseteq X$.

So, inequalities (4.2) and (4.3) are proven.

From them it follows by induction that

$$\tilde{\rho}^A(\omega_X) \leqslant \exp\left\{-\beta \sum_{x \in X} (1 - \|\Psi\|_1 - 3\varepsilon)\, v(\omega_x) + F\right\}, \tag{4.10}$$

where

$$F = \ln\left(1 + C' + \sum_{s \geqslant P} e^{-\beta\varepsilon\psi_{s+1} V_{s+1} + D'' V_{s+1}}\right). \tag{4.11}$$

Indeed, let's assume that (4.10) holds for $X' \subset X$ then (4.2) and (4.3) result in (F is positive, $|[P+1]| > 1$)

$$\begin{aligned}
\tilde{\rho}^A(\omega_X) &= \tilde{\rho}'(\omega_X) + \tilde{\rho}''(\omega_X) \\
&\leqslant \exp\left\{-\sum_{x \in X} \beta(1 - \|\Psi\|_1 - 3\varepsilon)\, v(\omega_x)\right\} \\
&\quad \times \left[C' e^{(|X|-1)F} + \sum_{s \geqslant P} e^{-\beta\varepsilon\psi_{s+1} V_{s+1} + D'' V_{s+1}} e^{|X \setminus [s+1]|\, F}\right] \\
&\leqslant \exp\left\{-\beta \sum_{x \in X} (1 - \|\Psi\|_1 - 3\varepsilon)\, v(\omega_x)\right\} e^{(|X|-1)F} \\
&\quad \times \left[C' + \sum_{s \geqslant P} e^{-\beta\varepsilon\psi_{s+1} V_{s+1} + D'' V_{s+1}} e^{(1 - |[s+1]|)\, F}\right] \\
&\leqslant \exp\left\{-\beta \sum_{x \in X} (1 - \|\Psi\|_1 - 3\varepsilon)\, v(\omega_x)\right\} e^{(|X|-1)F} \\
&\quad \times \left[C' + \sum_{s \geqslant P} e^{-\beta\varepsilon\psi_{s+1} V_{s+1} + D'' V_{s+1}}\right].
\end{aligned}$$

(4.10) yields the needed superstability bound

$$\tilde{\rho}^A(\omega_X) \leqslant \exp\left\{-\sum_{x \in X} [\beta(1 - 3\varepsilon)\, v(\omega_x) - c_0]\right\}, \qquad c_0 = \|\Psi\|_1\, \psi_P V_P + F. \tag{4.12}$$

Indeed, let's put at first $v(\omega_x) \leqslant \psi_P V_P$ for each $x \in X$. Then (4.12) follows immediately from (4.10).

If for some x, $v(\omega_x) \geqslant \psi_P V_P$, then putting $x = 0$ (by translation), using (4.3) ($\tilde{\rho}' = 0$) and induction we derive assuming (4.12) holds for $X' \subset X$ ($c_0 > 0$, $|[P+1]| > 1$)

$$\tilde{\rho}^A(\omega_X) \leqslant \sum_{s\geqslant P} \exp\left\{-\beta \sum_{x\in X} (1-3\varepsilon)\, v(\omega_x)\right\} e^{c_0\,|X\setminus[s+1]|} e^{-\beta\varepsilon\psi_{s+1}V_{s+1}+D''V_{s+1}}$$

$$\leqslant \exp\left\{-\sum_{x\in X} \beta(1-3\varepsilon)\, v(\omega_x)\right\} e^{(|X|-1)\,c_0} \sum_{s\geqslant P} e^{c_0(1-\|[s+1]\|)} e^{-\beta\varepsilon\psi_{s+1}V_{s+1}+D''V_{s+1}}$$

$$\leqslant \exp\left\{-\sum_{x\in X} \beta(1-3\varepsilon)\, v(\omega_x)\right\} e^{c_0(|X|-1)+F}.$$

To prove (2.5) and (2.6) we have to take into account (4.10)–(4.12), Remark 4.1, put $\xi = e^{\beta D'}$, $c^0 = \|\Psi\|_1\, V_P \psi_P$ and apply the bound $(1+I)^{V_j} \leqslant (2I)^{V_j}$, where I is the integral in the expression for D''.

Proofs of Lemma 4.1 and Proposition 4.1 are obtained by standard arguments from ref. 11 which have to take into account the following generalization of Lemma 2.3 from ref. 11.

Proposition 4.2. If the conditions of Theorem 2.2 are satisfied then

(a) $$\frac{\psi_{j+1}V_{j+1}}{\psi_j V_j} \leqslant (1+2\alpha)^{d+k} \leqslant (1+3\alpha)^{d+k}$$

(b) $$\frac{\psi_{j+2}V_{j+2}-\psi_j V_j}{\psi_j V_j} \leqslant (1+2\alpha)^{2(d+k)}-1 \leqslant (1+3\alpha)^{2(d+k)}-1,$$

(c) $$\frac{\psi_{q+s+2}V_{q+s+2}}{\psi(l_{q+s+1}-l_{q+1}+1)(2l_{q+s+1}-2l_{q+1}+3)^d} \leqslant ((1+2\alpha)(1+\alpha)\,\alpha^{-1})^{k+d}.$$
$$\leqslant ((1+3\alpha)(1+\alpha)\,\alpha^{-1})^{k+d}.$$

The proofs (a) and (b) are quite obvious since

$$\frac{V_{j+1}}{V_j} \leqslant \left(\frac{l_{j+1}}{l_j}\right)^d = (1+2\alpha)^d$$

In the proof of (c) the following inequalities are exploited

$$\frac{l^k_{q+s+2}}{l^k_{q+s+1}-l^k_{q+1}+1} \leqslant \left(\frac{l_{q+s+2}}{l_{q+s+1}-l_{q+1}}\right)^k = (1+2\alpha)^k\,(1-(1+2\alpha)^{-s})^{-k}$$
$$\leqslant (1+2\alpha)^k\,(1-(1+\alpha)^{-1})^{-k} = ((1+2\alpha)(1+\alpha)\,\alpha^{-1})^k,$$

$$\frac{(1+2l_{q+s+2})^d}{(2l_{q+s+1}-2l_{q+1}+3)^d} \leqslant \frac{l^d_{q+s+2}}{(l_{q+s+1}-l_{q+1}+1)^d} \leqslant \frac{l^d_{q+s+2}}{(l_{q+s+1}-l_{q+1})^d}$$
$$\leqslant ((1+2\alpha)(1+\alpha)\,\alpha^{-1})^d.$$

Proof of (4.5). From the defintion of U_-, W_-, W^+ we derive

$$U(\omega_\Lambda)-U^+(\omega_X)=U_-(\omega_\Lambda)+U^+(\omega_{\Lambda\setminus X})+W^+(\omega_X;\omega_{\Lambda\setminus X})$$
$$=U_-(\omega_x)+U_-(\omega_{\Lambda\setminus x})+W_-(\omega_x;\omega_{\Lambda\setminus x})+U^+(\omega_{\Lambda\setminus X})$$
$$+W^+(\omega_X;\omega_{\Lambda\setminus X}).$$

The equality leads to

$$U(\omega_\Lambda)-U^+(\omega_X)-W_-(\omega_x;\omega_{\Lambda\setminus x})+U(\omega'_x)$$
$$=U_-(\omega_x)+U_-(\omega'_x,\omega_{\Lambda\setminus x})+U^+(\omega'_x,\omega_{\Lambda\setminus X})-W_-(\omega'_x;\omega_{\Lambda\setminus x})$$
$$-W^+(\omega'_x;\omega_{\Lambda\setminus X})+W^+(\omega_X;\omega_{\Lambda\setminus X})$$
$$\geqslant U_-(\omega_x)+U_-(\omega'_x,\omega_{\Lambda\setminus x})+U^+(\omega'_x,\omega_{\Lambda\setminus X})$$
$$-W_-(\omega'_x;\omega_{\Lambda\setminus x})-W^+(\omega'_x;\omega_{\Lambda\setminus X})+W^+(\omega_{X\setminus x};\omega_{\Lambda\setminus X})$$
$$=U_-(\omega_x)+U_-(\omega'_x,\omega_{\Lambda\setminus x})+U^+(\omega'_x,\omega_{\Lambda\setminus X})-W_-(\omega'_x;\omega_{\Lambda\setminus x})$$
$$-W^+(\omega'_x;\omega_{\Lambda\setminus x})+W^+(\omega_{X\setminus x};\omega'_x,\omega_{\Lambda\setminus X}).$$

On the first step (second line)we merely used the definition of U_-, W_-, W^+ and the equality $U^+(\omega)=0$. On the second step we applied the inequality

$$W^+(\omega_X;\omega_{\Lambda\setminus X})\geqslant W^+(\omega_{X\setminus x};\omega_{\Lambda\setminus X})$$

and on the third step we used the equality

$$W^+(\omega_{X\setminus x};\omega'_x,\omega_{\Lambda\setminus X})=W^+(\omega_{X\setminus x};\omega_{\Lambda\setminus X})+W^+(\omega_{X\setminus x};\omega'_x)$$

(4.5) follows from the equality

$$U^+(\omega'_x,\omega_{\Lambda\setminus x})-U^+(\omega_{X\setminus x})=U^+(\omega'_x,\omega_{\Lambda\setminus X})+W^+(\omega_{X\setminus x};\omega'_x,\omega_{\Lambda\setminus X}).$$

Proof of (4.6). Let $[s+1]\subseteq X$. For this case we merely repeat the above arguments. From the definition of U_-, W_-, W^+ we derive

$$U(\omega_\Lambda)-U^+(\omega_X)=U_-(\omega_\Lambda)+U^+(\omega_{\Lambda\setminus X})+W^+(\omega_X;\omega_{\Lambda\setminus X})$$
$$=U_-(\omega_{[s+1]})+U_-(\omega_{\Lambda\setminus[s+1]})+W_-(\omega_{[s+1]};\omega_{\Lambda\setminus[s+1]})$$
$$+U^+(\omega_{\Lambda\setminus X})+W^+(\omega_X;\omega_{\Lambda\setminus X}).$$

The equality leads to

$$\begin{aligned}U(\omega_\Lambda)-U^+(\omega_X)-W_-(\omega_{[s+1]};\omega_{\Lambda\setminus[s+1]})+U(\omega'_{[s+1]})\\
&=U_-(\omega_{[s+1]})+U_-(\omega'_{[s+1]},\omega_{\Lambda\setminus[s+1]})+U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus X})\\
&\quad-W_-(\omega'_{[s+1]};\omega_{\Lambda\setminus[s+1]})-W^+(\omega'_{[s+1]};\omega_{\Lambda\setminus X})+W^+(\omega_{X\setminus[s+1]};\omega_{\Lambda\setminus X})\\
&\geqslant U_-(\omega_{[s+1]})+U_-(\omega'_{[s+1]},\omega_{\Lambda\setminus[s+1]})+U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus X})\\
&\quad-W_-(\omega'_{[s+1]};\omega_{X\setminus[s+1]})-W^+(\omega'_{X\setminus[s+1]};\omega_{\Lambda\setminus[s+1]})\\
&\quad+W^+(\omega_{X\setminus[s+1]};\omega'_{[s+1]},\omega_{\Lambda\setminus X}).\end{aligned}$$

Here we used the definition of U_-, W_-, W^+ , the inequality

$$W^+(\omega_X;\omega_{\Lambda\setminus X})\geqslant W^+(\omega_{X\setminus[s+1]};\omega_{\Lambda\setminus X}),$$

the equality

$$W^+(\omega_{X\setminus[s+1]};\omega'_{[s+1]},\omega_{\Lambda\setminus X})=W^+(\omega_{X\setminus[s+1]};\omega_{\Lambda\setminus X})+W^+(\omega_{X\setminus[s+1]};\omega'_{[s+1]})$$

and the relation $(\omega_{X\setminus[s+1]},\omega_{\Lambda\setminus X})=\omega_{\Lambda\setminus[s+1]}$. To obtain (4.6) is only necessary to take into account the equality

$$\begin{aligned}U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus[s+1]})-U^+(\omega_{X\setminus[s+1]})\\
&=U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus X})+W^+(\omega_{X\setminus[s+1]};\omega'_{[s+1]},\omega_{\Lambda\setminus X}).\end{aligned}$$

Let $[s+1]\setminus[s+1]\cap X\neq\varnothing$. By the similar arguments as above we derive

$$\begin{aligned}U(\omega_\Lambda)-U^+(\omega_X)\\
&=U_-(\omega_\Lambda)+U^+(\omega_{\Lambda\setminus X})+W^+(\omega_X;\omega_{\Lambda\setminus X})\\
&=U_-(\omega_{[s+1]})+U_-(\omega_{\Lambda\setminus[s+1]})+W_-(\omega_{[s+1]};\omega_{\Lambda\setminus[s+1]})\\
&\quad+U^+(\omega_{\Lambda\setminus X})+W^+(\omega_X;\omega_{\Lambda\setminus X})\geqslant U_-(\omega_{[s+1]})+U_-(\omega_{\Lambda\setminus[s+1]})\\
&\quad+W_-(\omega_{[s+1]};\omega_{\Lambda\setminus[s+1]})+U^+(\omega_{\Lambda\setminus([s+1]\cup X)})+W^+(\omega_{X\setminus[s+1]};\omega_{\Lambda\setminus([s+1]\cup X)}).\end{aligned}$$

The inequality leads to

$$\begin{aligned}U(\omega_\Lambda)-U^+(\omega_X)-W_-(\omega_{[s+1]};\omega_{\Lambda\setminus[s+1]})+U(\omega'_{[s+1]})\\
&\geqslant U_-(\omega_{[s+1]})+U_-(\omega'_{[s+1]},\omega_{\Lambda\setminus[s+1]})-W_-(\omega'_{[s+1]};\omega_{\Lambda\setminus[s+1]})\\
&\quad+U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus([s+1]\cup X)})-W^+(\omega'_{[s+1]};\omega_{\Lambda\setminus([s+1]\cup X)})\\
&\quad+W^+(\omega_{X\setminus[s+1]};\omega_{\Lambda\setminus([s+1]\cup X)})=U_-(\omega_{[s+1]})+U_-(\omega'_{[s+1]},\omega_{\Lambda\setminus[s+1]})\\
&\quad-W_-(\omega'_{[s+1]};\omega_{\Lambda\setminus[s+1]})+U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus([s+1]\cup X)})\\
&\quad-W^+(\omega'_{[s+1]};\omega_{X\setminus[s+1]},\omega_{\Lambda\setminus([s+1]\cup X)})+W^+(\omega_{X\setminus[s+1]};\omega'_{[s+1]},\omega_{\Lambda\setminus([s+1]\cup X)}).\end{aligned}$$

Now we have to account the equality

$$U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus[s+1]})-U^+(\omega_{X\setminus[s+1]})$$
$$=U^+(\omega'_{[s+1]},\omega_{\Lambda\setminus([s+1]\cup X)})+W^+(\omega_{X\setminus[s+1]};\omega'_{[s+1]},\omega_{\Lambda\setminus([s+1]\cup X)}).$$

and the relation

$$\Lambda\setminus[s+1]=X\setminus[s+1]\cup\Lambda\setminus([s+1]\cup X).$$

(4.6) is proven. This concludes the proof of Theorem 2.2.

5. PEIERLS ARGUMENT AND ASYMPTOTICS OF *f*

We start the section from proving Lemma 1.1.

The set of all configurations q_Λ can be described by the set of configurations s_Λ, as in the Ising model. The set of all spin configurations can be classified by different contours $\gamma(s_\Lambda)$, i.e., connected union of faces of unit cubes, centered at lattice sites, which is a boundary of a related connected union of the cubes. The main idea is to consider contours $\gamma_{x,y}$, enclosing x, separating it from y and with adjacent cubes, containing spins of different signs from the opposite sides. So inside $\gamma_{x,y}$ there are spins of both signs. The contours may be non-closed ending on the boundary $\partial\Lambda$. There may be several such the contours in a configuration. In this case the smallest contour is chosen. We have to estimate the l.h.s. of (1.9) in terms of such the contours. With this aim we express it as a sum over s_Λ and then transform this sum into the sum over the contours $\gamma_{x,y}\in\Lambda$, summing over all configurations, characterized by the contours. So, at first we have to insert the equality

$$1=\prod_{l\in\Lambda}(\chi_l^+ +\chi_l^-)$$

under the sign of the Gibbs average. As a result

$$\langle\chi_x^+\chi_y^-\rangle_\Lambda=\sum_{s_\Lambda}\left\langle\chi_x^+\chi_y^-\prod_{l\in\Lambda}\chi_l^{s_l}\right\rangle_\Lambda$$
$$=\sum_{\gamma_{x,y}\in\Lambda}\;\sum_{s_\Lambda:\,\gamma(s_\Lambda)=\gamma_{x,y}\in\Lambda}\left\langle\chi_x^+\chi_y^-\prod_{l\in\Lambda}\chi_l^{s_l}\right\rangle_\Lambda$$
$$\leqslant\sum_{\gamma_{x,y}\in\Lambda}\left\langle\chi_x^+\chi_y^-\prod_{x,x'\in N_\gamma}\chi_x^+\chi_{x'}^-\right\rangle_\Lambda.$$

Hence

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda \leqslant \sum_{\gamma_{x,y} \in \Lambda} \left\langle \prod_{x, x' \in N_\gamma} \chi_x^+ \chi_{x'}^- \right\rangle_\Lambda$$

Enlarging the range of the summation to all closed contours γ we obtain

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda \leqslant \sum_{\gamma} \left\langle \prod_{x, x' \in N_\gamma} \chi_x^+ \chi_{x'}^- \right\rangle_\Lambda$$

From (1.8) it follows that

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda \leqslant \sum_{\gamma} e^{-E\,|N_\gamma|} \leqslant \sum_{\gamma} e^{-2E\,|\gamma|}$$

$$= \sum_{n_1 \geqslant 1, \ldots, n_d \geqslant 1} e^{-2E(\sum_{s=1}^d n_s)} \left(\sum_{\gamma:\, |\gamma|_s = n_s} 1 \right) = (2^{\frac{1}{d-1}} 3)^{-d} \left(\sum_{n \geqslant 1} (3e^{-2E})^n \, n^{\frac{1}{d-1}} \right)^d,$$

where $|\gamma|_s$ is a number of faces of γ, orthogonal to sth coordinate axis. Here we used the inequalities $|\gamma| = n_1 + \cdots + n_d$,

$$\sum_{\gamma:\, |\gamma|_s = n_s} 1 \leqslant \prod_{s=1}^d 3^{n_s - 1} \, |\mathrm{int}(\gamma)|, \qquad |\mathrm{int}(\gamma)| \leqslant \prod_{s=1}^d \left(\frac{n_s}{2} \right)^{\frac{1}{d-1}},$$

where $\mathrm{int}(\gamma)$ is the set of lattice sites inside γ (see ref. 17, paragraph 5.3, Lemmas 5.3.5 and 5.3.6). Lemma (1.1) is true if $e^{-2E} \leqslant \frac{1}{6}$ and in this case

$$a' = 3(2^{\frac{1}{d-1}} 3)^{-d} \left(\sum_{n \geqslant 1} 6^{-n+1} n^{\frac{1}{d-1}} \right)^d, \qquad a = 2d.$$

This conclude the proof of the lemma.

(1.11) follows from the inequality

$$\chi^+(q_x)\, \chi^-(q_y) \leqslant \exp\{Q(q_x, q_y) - cR^2\}, \qquad R, c > 0, \tag{5.1}$$

where

$$Q(q_x, q_y) = c[(q_x - q_y)^2 + \tfrac{4}{3}(|q_x^2 - R^2| + |q_y^2 - R^2|)]$$

It is derived easily from two inequalities by putting $R = e_0$, $c = e_0^{-1}$.

$$\chi^+(q_x)\, \chi^-(q_y) \leqslant e^{-c[R^2 - (q_x - q_y)^2]}, \qquad |q_x|, |q_y| \geqslant 2^{-1} R,$$

$$\chi^+(q_x)\, \chi^-(q_y) \leqslant e^{-c[R^2 - \frac{4}{3}(|q_x^2 - R^2| + |q_y^2 - R^2|)]}, \qquad |q_x|, |q_y| \leqslant 2^{-1} R.$$

For $|q_x| \leqslant \frac{R}{2}$, $|q_y| \geqslant \frac{R}{2}$ the second term in the expression for Q is not less than cR^2.

Proof of Corollary 2.1. We have to establish the asymptotic behavior of f. We have

$$f(\varepsilon, z) = \sum_{j \geqslant 0} e^{-\varepsilon l_j^k (1+2l_j)^d} (2z)^{(1+2l_j)^d}. \tag{5.2}$$

Let's put $z_0 = 2^{2d - \frac{d^2}{k+d}} |\ln 2z|$, $L_j = (2^d \varepsilon)^{\frac{1}{k+d}} l_j$. Then, applying the bound $(1+2l_j)^d \leqslant 2^d(1+(2l_j)^d)$, we obtain

$$\begin{aligned} f(\varepsilon, z) &\leqslant e^{2^d |\ln 2z|} \sum_{j \geqslant 0} e^{-\varepsilon 2^d l_j^{k+d} + 4^d l_j^d |\ln 2z|} \\ &= e^{2^d |\ln 2z|} \sum_{j \geqslant 0} e^{-L_j^{k+d} + \varepsilon^{-\frac{d}{k+d}} L_j^d z_0}, \end{aligned}$$

$$\sum_{L_j \leqslant 1} e^{-L_j^{k+d} + \varepsilon^{-\frac{d}{k+d}} L_j^d z_0} \leqslant e^{\varepsilon^{-\frac{d}{k+d}} z_0} \sum_{L_j \leqslant 1} 1.$$

For $k \geqslant d$

$$\begin{aligned} \sum_{L_j \geqslant 1} e^{-L_j^{k+d} + \varepsilon^{-\frac{d}{k+d}} L_j^d z_0} &\leqslant e^{4^{-1} \varepsilon^{-\frac{2d}{k+d}} z_0^2} \sum_{j \geqslant 0} e^{-(L_j^d - 2^{-1} \varepsilon^{-\frac{d}{k+d}} z_0)^2} \\ &= e^{4^{-1} \varepsilon^{-\frac{2d}{k+d}} z_0^2} \sum_{j \geqslant 0} e^{-(2^d \varepsilon)^{\frac{2d}{k+d}} (l_j^d - 2^{-1} (2^d \varepsilon^2)^{-\frac{d}{k+d}} z_0)^2} \end{aligned} \tag{5.3}$$

Let's determine ε (see the condition of Theorem 2.1) from the following inequality

$$(1+3\alpha)^{2(d+k)} - 1 \leqslant (2+4\alpha)^{2(d+k)} - 2^{2(d+k)} = \|\Psi\|_1^{-1} 2^{-1} \varepsilon,$$

and put

$$(1+2\alpha)^{2(d+k)} - 1 = \varepsilon', \qquad \varepsilon' = 2^{-2(d+k)-1} \|\Psi\|_1^{-1} \varepsilon.$$

As a result

$$l_j^d = (1+2\alpha)^{dj} = (1+\varepsilon')^{\frac{dj}{2(d+k)}}.$$

If $L_j \leqslant 1$ then $l_j \leqslant (2^d \varepsilon)^{-\frac{1}{k+d}}$ and $(1+\varepsilon')^{\frac{j}{2(d+k)}} \leqslant (2^d \varepsilon)^{-\frac{1}{k+d}}$. So, if $L_j \leqslant 1$ then

$$j \leqslant 2(\ln(1+\varepsilon'))^{-1} |\ln 2^d \varepsilon| \leqslant 2(\varepsilon')^{-1} |\ln 2^d \varepsilon|.$$

As a result

$$\sum_{L_j \leqslant 1} e^{-L_j^{k+d} + \varepsilon^{-\frac{d}{k+d}} L_j^d z_0} \leqslant \|\Psi\|_1 \, 2^{2+2(d+k)} e^{\varepsilon^{-\frac{d}{k+d}} z_0} \varepsilon^{-1} \, |\ln 2^d \varepsilon| \tag{5.4}$$

Now, let's consider the sum in which $L_j \geqslant 1$. For $\varepsilon' \leqslant 1$ and $dj \geqslant d+k$ we have $l_j^d \geqslant (1+\varepsilon')^{[\frac{dj}{2(d+k)}]} \geqslant 1 + [\frac{dj}{2(d+k)}] \, \varepsilon' \geqslant \frac{dj}{2(d+k)} \, \varepsilon'$, where $[\frac{dj}{2(d+k)}]$ is the integer part of the number.

Let's decompose the set $\{j \geqslant 1 + \frac{k}{d}\}$ of summation into the set G and its compliment G^c,

$$G = \left\{ j \colon j \leqslant \frac{d+k}{d} \, (\varepsilon')^{-1} \, (2^d \varepsilon^2)^{-\frac{d}{k+d}} \, z_0 \right\},$$

As a result for $k \geqslant d$ we have

$$\begin{aligned}
&\sum_{j \geqslant 0} e^{-(2^d \varepsilon)^{\frac{2d}{k+d}} (l_j^d - 2^{-1} (2^d \varepsilon^2)^{-\frac{d}{k+d}} z_0)^2} \\
&\quad \leqslant |G| + \sum_{j \leqslant 1 + \frac{k}{d}} 1 + \sum_{j \in G^c} e^{-(2^d \varepsilon)^{\frac{2d}{k+d}} (j \varepsilon' \frac{d}{2(d+k)} - 2^{-1} (2^d \varepsilon^2)^{-\frac{d}{k+d}} z_0)^2} \\
&\quad \leqslant |G| + \left(1 + \frac{k}{d}\right) + 2 \sum_{j \geqslant 0} e^{-(2\varepsilon)^{\frac{2d}{k+d}} j^2 (\varepsilon' \frac{d}{4(d+k)})^2} \\
&\quad \leqslant |G| + \left(1 + \frac{k}{d}\right) + 2 \sum_{j \geqslant 0} e^{-(2\varepsilon)^{\frac{2d}{k+d}} j (\varepsilon' \frac{d}{4(d+k)})^2} \\
&\quad = |G| + \left(1 + \frac{k}{d}\right) + 2 (1 - e^{-(2\varepsilon)^{\frac{2d}{k+d}} (\varepsilon' \frac{d}{4(d+k)})^2})^{-1}.
\end{aligned}$$

This expression grows as $\varepsilon^{-2-\frac{2d}{k+d}}$. (5.4-5) prove the first inequality in Corollary 2.1. The second one follows from (a) and and the last statement of Theorem 2.2.

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